LESSON

# 5 Limits

Mathematicians, just like children, like to see what happens when we push limits. We are told not to divide by zero, so the temptation overwhelms us to see what happens when we divide by *almost* zero. The process of using *almost* numbers underlies the concept of a limit.

Limits can be most easily seen graphically. For example, look at the graph of $y = f(x)$ in Figure 5.1.

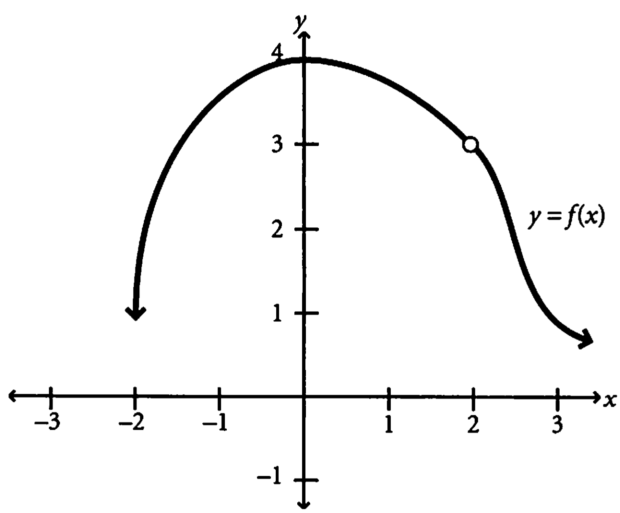



**Figure 5.1**

The domain of $f$ is $x \neq 2$. We can't plug $x = 2$ into $f$. However, the hole is in a clear place, at (2,3). How do we know the hole has a $y$-value of 3? Well, the points on the curve with $x$-values *near* $x = 2$ have $y$-values *close* to $y = 3$. The closer we get to $x = 2$, the closer the $y$-values of the points come to $y = 3$.

The mathematical shorthand for this is $\lim_{x\to 2} f(x) = 3$, which is pronounced "the limit as $x$ approaches 2 of $f(x)$ is 3."

We don't have to just approach discontinuities, though. For example, $\lim_{x\to 0} f(x) = 4$. Note that this is a statement about the points *near* $x = 0$ having $f(x)$ *near* 4. The exact point at (0,4) isn't used in evaluating the limit.

We can also approach points from either the left or from the right. For example, take Figure 5.2 to be the graph of $y = g(x)$.

Here, $\lim_{x\to 1^-} g(x) = 4$ and $\lim_{x\to 1^+} g(x) = 2$. The little minus in $\lim_{x\to 1^+}$ means that we approach $x = 1$ using numbers less than (to the left) of $x = 1$. As we approach $x = 1$ from the left-hand side, we slide up the graph through $y$-values that approach 4. Similarly, the plus in $\lim_{x\to 1^+}$ means "approach from the right." From the right, the height of the graph slides down to $y = 2$ as $x$ approaches 1.

In this example, $\lim_{x\to 1} g(x)$ does not exist because there is no single $y$-value to which all of the points near $x = 1$ get close. Some are close to 4, and others are close to 2. Because there is no agreement, there is no limit.

As another example, let Figure 5.3 be the graph of $y = h(x)$. Here, $\lim_{x\to 3^-} h(x) = 2$ because sliding up to $x = 3$ from the left has us pass through points with $y$-values near 2. Similarly, $\lim_{x\to 3^+} h(x) = 2$. Because there *is* agreement from the left and right, we have the general limit, $\lim_{x\to 3} h(x) = 2$. Once again, notice that what happens at exactly $x = 3$ is irrelevant. Here $h(3) = 5$, but the resulting point at (3,5) has no bearing on the limit of points *approaching* $x = 3$.

Vertical asymptotes correspond with infinite limits. For example, take the graph in Figure 5.4 of $y = k(x)$.

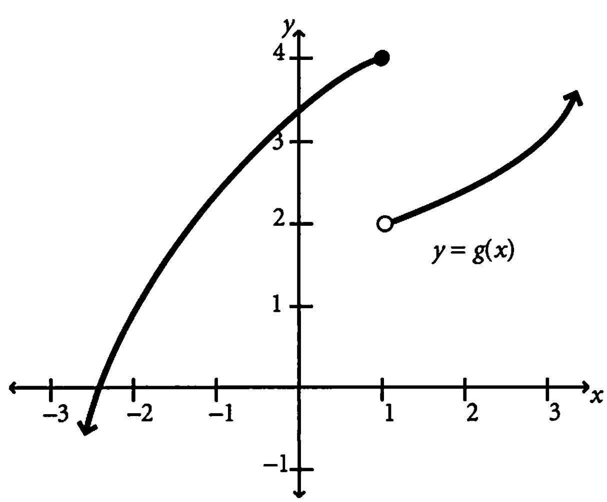



**Figure 5.2**

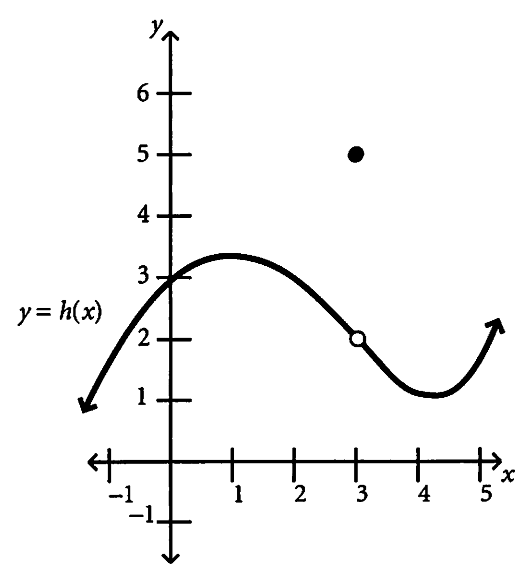



**Figure 5.3**

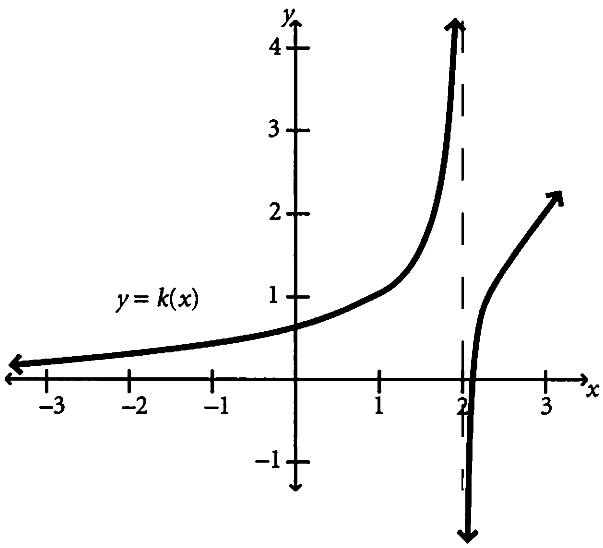



**Figure 5.4**

Here, $\lim\limits_{x\to 2^-} k(x) = \infty$, $\lim\limits_{x\to 2^+} k(x) = -\infty$, and $\lim\limits_{x\to 2} k(x)$ does not exist.

All of these examples involve discontinuities. We can rule them out in the following manner. If $\lim\limits_{x\to a^-} f(x) = f(a) = \lim\limits_{x\to a^+} f(x)$, then $f$ is continuous at $x = a$. In other words, if $f(a)$ exists (there is a point at $x = a$) and the limit from either side goes up to the same value, then the function flows continuously through that point.

## ▶ Practice

Use Figure 5.5 to evaluate the following.

1. $\lim\limits_{x\to -1^-} f(x)$

2. $\lim\limits_{x\to -1^+} f(x)$

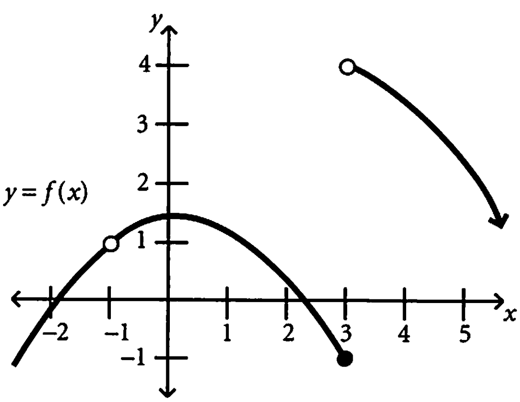
3. $\lim\limits_{x\to -1} f(x)$

4. $f(-1)$

5. Is $f$ continuous at $x = -1$?

6. $\lim\limits_{x\to 3^-} f(x)$

7. $\lim\limits_{x\to 3^+} f(x)$



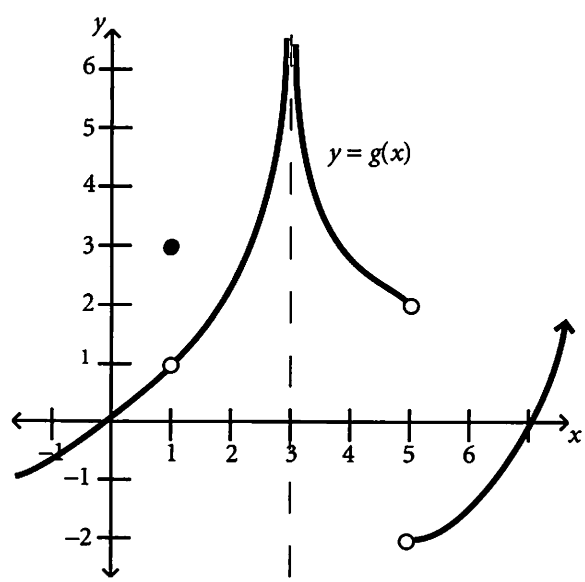




**Figure 5.5**

8. $\lim_{x \to 3} f(x)$

9. $f(3)$

10. Is $f$ continuous at $x = 1$?

11. $\lim_{x \to 1} g(x)$

12. $g(1)$

13. $\lim_{x \to 3^-} g(x)$

14. $\lim_{x \to 3} g(x)$

15. $\lim_{x \to 5^-} g(x)$

16. $\lim_{x \to 5^+} g(x)$

## ▶ Evaluating Limits Algebraically

It is not necessary to have the graph of a function to evaluate its limits. If the limit can be plugged in without dividing by zero, that is how the limit is calculated.

Technically, this works only with functions that are *polynomials* (formed by a variable added and multiplied with constants) like $4x^5 - 10x^3 - 7$, roots like $\sqrt{x}$, *rational functions* (formed by dividing two polynomials) like $\frac{3x-5}{2x^3+x^2+1}$, and *transcendental functions* like the trigonometric functions, $\ln(x)$, and $e^x$. Because this works for any combination of these functions added, subtracted, multiplied, divided, or composed, it works also for every function considered in this book. The reason is that all of these functions are continuous on their domains, and continuity ensures that the limits approach the points obtained by plugging in values.

### Example

Evaluate $\lim_{x \to 4} \frac{x+5}{x^2+10x}$ and $\lim_{x \to -2} 3x^2 + x - 7$.

### Solution

Because 4 can be plugged into $\frac{x+5}{x^2+10x}$ without there being a division by zero, the limit $\lim_{x \to 4} \frac{x+5}{x^2+10x}$ $= \frac{4+5}{16+40} = \frac{9}{56}$. Similarly, $\lim_{x \to -2} 3x^2 + x - 7 =$ $3(-2)^2 + (-2) - 7 = 3$.

## ▶ Practice

Evaluate the following limits.

17. $\lim_{x \to 1} 10x^3 + 4x^2 - 5x + 7$

18. $\lim_{x \to 2} \frac{x^3 - 4}{10x + 3}$

19. $\lim_{x \to 3} \frac{x - 3}{x^2 + x}$

20. $\lim_{x \to \frac{\pi}{6}} \frac{\sin(x)}{x}$

21. $\lim_{a \to 0} 2x + a + 1$

22. $\lim_{a \to 0} 3x^2 + 3xa + a^2$

Dividing by a tiny number is equivalent to multiplying by an enormous number. For example:

$$5 \div \frac{1}{10{,}000} = 5 \cdot \frac{10{,}000}{1} = 50{,}000$$

It is for this reason that if the denominator of a fraction approaches zero while the numerator goes to something nonzero, the result is an infinite limit.

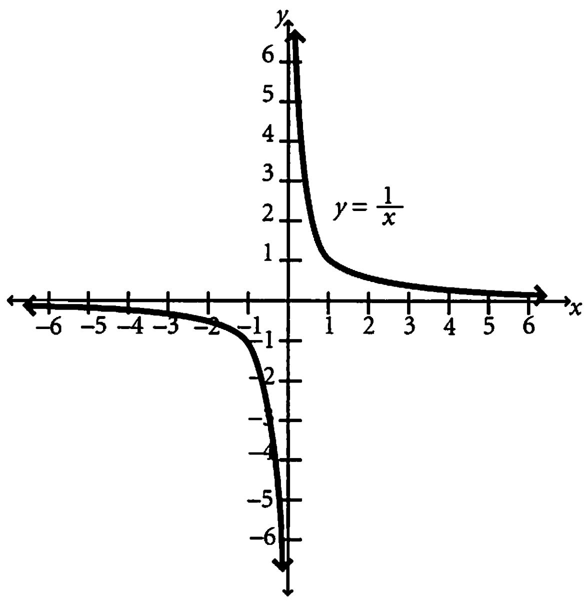



**Figure 5.6**

The classic example is $f(x) = \frac{1}{x}$ (graphed in Figure 5.6). This has the following limits at zero: $\lim_{x\to0^-}\frac{1}{x} = -\infty$, $\lim_{x\to0^+}\frac{1}{x} = \infty$, and $\lim_{x\to0}\frac{1}{x}$ does not exist.

Because the denominator goes to zero while the numerator stays one in all of these cases, there is a vertical asymptote at $x = 0$. The function therefore approaches either positive or negative infinity from either side. When $x$ is less than zero, as it always is when $x\to0^-$, the function $\frac{1}{x}$ is also negative. Thus, $\lim_{x\to0^-}\frac{1}{x} = -\infty$. Similarly, as $x\to0^+$, $\frac{1}{x}$ is always positive, so $\lim_{x\to0^+}\frac{1}{x} = \infty$. Finally, because the limit from the two sides are different, the undirected limit $\lim_{x\to0}\frac{1}{x}$ does not exist.

### Example

Evaluate $\lim_{x\to2}\frac{x+3}{(x-2)(x-4)}$.

### Solution

The numerator approaches 5 while the denominator approaches 0. Therefore, this limit from the right is either $\infty$ or $-\infty$. What we need to figure out is whether the function is positive or negative at $x$-values just slightly larger than 2. We do this by looking at each factor individually.

As $x\to2^+$, the factor $(x+3)\to5^+$ (a positive number), $(x-2)\to0^+$ (a positive number), and $(x-4)\to-2^+$ (a negative number). Because the function $\frac{x+3}{(x-2)(x-4)}$ is made of two positive parts and one negative part, the result will be negative. Thus, $\lim_{x\to2^+}\frac{x+3}{(x-2)(x-4)} = -\infty$.

There are other, perhaps easier, ways to evaluate such limits. One is to plug into the function a representative number. In the previous example, for instance, when $x = 2.01$, the function is $\frac{(2.01)+3}{((2.01)-2)((2.01)-4)} \approx -251$. Because this is negative, the limit is $-\infty$. Another method will be covered in Lesson 13.

### Example

Evaluate $\lim_{x\to-3^-}\frac{(x+1)(2-x)}{(x+3)(x+5)}$.

### Solution

Here, the numerator approaches $-10$, which isn't zero, while the denominator approaches zero, so the limit is either $\infty$ or $-\infty$. While $x\to-3^-$, the factors:

$(x+1)\to-2^-$ (negative)

$(2 - x) \to 5^+$ (positive)

$(x + 3) \to 0^-$ (negative)

$(x + 5) \to 2^-$ (positive)

The combination of two negative factors and two positive factors is positive, thus:

$$\lim_{x \to -3^-} \frac{(x + 1)(2 - x)}{(x + 3)(x + 5)} = \infty$$

## ► Practice

Evaluate the following limits.

**23.** $\lim_{x \to 1^-} \frac{1}{x - 1}$

**24.** $\lim_{x \to 4^+} \frac{x + 5}{x - 4}$

**25.** $\lim_{x \to 3^+} \frac{x - 2}{x + 3}$

**26.** $\lim_{x \to 3^+} \frac{(x + 2)(x - 5)}{(x + 6)(x - 3)}$

**27.** $\lim_{x \to 2} \frac{(x + 5)(x - 5)}{(x - 3)(x + 4)}$

**28.** $\lim_{x \to -5} \frac{x - 2}{(x + 5)^2}$

When both the numerator *and* the denominator go to zero, then there are two common tricks for simplifying the limit. The first is to factor. The second is to rationalize. The following example utilizes the first trick—factoring.

### Example

Evaluate $\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 + x - 20}$.

### Solution

Here, both the numerator and denominator go to zero, so we aren't guaranteed an infinite limit. First, factor the numerator and denominator.

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 + x - 20} = \lim_{x \to 4} \frac{(x - 4)(x + 2)}{(x - 4)(x + 5)}$$

Because $x \neq 4$ as $x \to 4$, we can cancel $\frac{x - 4}{x - 4} = 1$.

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 + x - 20} = \lim_{x \to 4} \frac{\cancel{(x - 4)}(x + 2)}{\cancel{(x - 4)}(x + 5)} = \lim_{x \to 4} \frac{(x + 2)}{(x + 5)}$$

Now we can plug in without dividing by zero.

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 + x - 20} = \lim_{x \to 4} \frac{(x + 2)}{(x + 5)} = \frac{6}{9} = \frac{2}{3}$$

The following example utilizes the trick of rationalizing.

### Example

Evaluate $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

### Solution

Because both numerator and denominator go to zero, a trick is necessary. First, multiply the top and bottom by the part with the square root, but with the opposite sign between them.

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \left(\frac{\sqrt{x} - 3}{x - 9}\right) \cdot \left(\frac{\sqrt{x} + 3}{\sqrt{x} + 3}\right)$$

Simplify.

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{x + 3\sqrt{x} - 3\sqrt{x} - 9}{(x - 9)(\sqrt{x} + 3)}$$

Eliminate $\frac{x - 9}{x - 9} = 1$.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

Plug in.

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)} = \frac{1}{6}$$

## ▶ Practice

Evaluate the following limits.

**29.** $\lim\limits_{x\to -2^+} \dfrac{(x-6)(x+2)}{(x+2)(x+1)}$

**30.** $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$

**31.** $\lim\limits_{x\to 4} \dfrac{x^2-9}{x+3}$

**32.** $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2+2x-15}$

**33.** $\lim\limits_{x\to 3^-} \dfrac{x+5}{x-3}$

**34.** $\lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{(x-25)(x+1)}$

**35.** $\lim\limits_{a\to 0} \dfrac{(x+a)^2-x^2}{a}$

**36.** $\lim\limits_{a\to 0} \dfrac{\sqrt{x+a}-\sqrt{x}}{a}$